Journal of the Indonesian Mathematical Society. 9(2): 37-39 (2003)

A Mathematical Model for Optimal Exploitation of Bio-Geographically Interconnected Natural Resources
(An Extended Abstract)

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Introduction

The exploitation of multi-species fisheries, such as fisheries which have biological interactions between species (population) or fisheries incorporating geographically interconnected species, is not well understood (Hilborn and Walters, 1992). Many scientists point out that appropriate policies for fisheries management are only possible if we have a comprehensive understanding of the underlying systems which are exploited (Yodzis, 1994; Botsford *et al*., 1997). They also argue that management practices could be improved if we include interactions between species in developing multi-species models (Hall, 1998).

In this paper we discuss a continuous mathematical model for the exploitation of a bio-geographically-interconnected populations. Biologically the population has a predator-prey interaction and geographically it has a metapopulation structure. The model here modifies Clark (1976) model for selective and explicit inshore-offshore model with the inclusion of two species interaction within two patches of habitat. These two patches are connected by the diffusion of the prey and the predator. The rates of the diffusion are assumed to be proportional to the differences of the population size between the two patches. It is assumed that individuals flow from the larger population size to the smaller. Let the number of prey and the predator in the patch *i* are denoted by N_i and P_i, respectively. If $N_2 > N_1$ and $P_2 > P_1$ then the prey and the predator move from patch two to patch one with the rate proportional to $(N_2 - N_1)$ and $(P_2 - P_1)$, respectively. It is also assumed that the only possible harvesting is a selective harvesting, in which we can choose how much effort that we can distribute to each population on each patch. Furthermore, in the absence of harvesting and diffusion, the dynamics of the prey and the predator on each patch are governed by the following structurally stable systems (May, 1973):

$$\frac{dN_i}{dt} = F_i(N_i, P_i) = r_i N_i \left(1 - \frac{N_i}{K_i}\right) + \alpha_i N_i P_i \tag{1}$$

$$\frac{dP_i}{dt} = G_i(N_i, P_i) = s_i P_i \left(1 - \frac{P_i}{L_i}\right) + \beta_i N_i P_i \tag{2}$$

where $r_i\ (s_i)$ and $K_i\ (L_i)$ denote the intrinsic growth rate of the prey (predator) and prey's (predator's) carrying capacity, respectively. In this case $\alpha_i < 0$ and $\beta_i > 0$ to ensure that the system reflects a predator-prey interaction. If symmetric diffusions and selective harvesting are introduced, then the complete model is:

$$\frac{dN_i}{dt} = F_i(N_i, P_i) + \sigma_N (N_j - N_i) - h_{Ni}(t) \tag{3}$$

$$\frac{dP_i}{dt} = G_i(N_i, P_i) + \sigma_P (P_j - P_i) - h_{Pi}(t) \tag{4}$$

where σ_N and σ_P are the coefficient of the rate of diffusion for the prey and the predator, respectively. The rate of harvesting for the prey and the predator on patch *i* are given by $h_{Ni}(t)$ and $h_{Pi}(t)$. These harvesting terms can be seen as production functions of the stock abundance. For simplicity we assume that $h_{Ni}(t) = E_{Ni} N_i$ and $h_{Pi}(t) = E_{Pi} P_i$, where $E_{Ni} N_i$ and $E_{Pi} P_i$ denote the amount of efforts to remove the harvests from prey population N_i and predator population P_i.

To obtain optimal harvests we use the concept of net present value maximization. In this case, we maximize the discounted net present value (*PV*) of the total revenue resulted from the harvests. If we choose $e^{-\delta}$ as a discounting factor, then the present value, which should be maximized, is given by

$$PV = \int_0^\infty e^{-\delta t} \sum_{i=1}^{2} \sum_{X \in \{N,P\}} R_{Xi}(X_i, h_{Xi}) dt \tag{5}$$

where R_{Ni} and R_{Pi} denote the net payment of the prey and the predator harvesting from patch *i*. We assume that the price of the stock do not depend on the location where they are harvested but only depend on whether they are prey or predator. On the other hand, the costs of harvesting are the function of the stocks and their locations. Furthermore, they will be assumed to be decreasing functions of the stock size. With these assumptions the function

$$R_{Xi} = \left(p_X - \frac{c_{Xi}}{X_i}\right) h_{Xi} \tag{6}$$

is a reasonable choice to reflect the net payment.

Using this equation, the total revenue of harvesting from all patches and from both prey and the predator is now given by

$$PV = \int_0^\infty e^{-\delta t} \sum_{i=1}^{2} \sum_{X \in \{N,P\}} \left(p_X - \frac{c_{Xi}}{X_i}\right) h_{Xi} dt \tag{7}$$

Hence, the objective of the manager of the resources is now to maximize the net present value in equation (7) subject to equations (3) and (4), assuming that $0 \le h_{Xi}(t) \le h_{Xi}$ max, $X_i(t) \ge 0$ and $X_i(0) = X_{i0}$. This maximization will produce implicit equations

$$\left(p_N - \frac{c_{N1}}{N_1}\right)\frac{\partial F_1}{\partial N_1} + \left(p_P - \frac{c_{P1}}{P_1}\right)\frac{\partial G_1}{\partial N_1} + \frac{c_{N1}}{N_1^2} F_1 = \left(p_N - \frac{c_{N1}}{N_1}\right)\delta + \sigma_N \left(\frac{c_{N2} N_1^2 - c_{N1} N_2^2}{N_1^2 N_2}\right) \tag{8}$$

$$\left(p_N - \frac{c_{N2}}{N2}\right)\frac{\partial F_2}{\partial N_2} + \left(p_P - \frac{c_{P2}}{P_2}\right)\frac{\partial G_2}{\partial N_2} + \frac{c_{N2}}{N_2^2} F_2 = \left(p_N - \frac{c_{N2}}{N_2}\right)\delta + \sigma_N \left(\frac{c_{N1} N_2^2 - c_{N2} N_1^2}{N_2^2 N_1}\right) \tag{9}$$

$$\left(p_P - \frac{c_{P1}}{P_1}\right)\frac{\partial G_1}{\partial P_1} + \left(p_N - \frac{c_{N1}}{N_1}\right)\frac{\partial F_1}{\partial P_1} + \frac{c_{P1}}{P_1^2} G_1 = \left(p_P - \frac{c_{P1}}{P_1}\right)\delta + \sigma_P \left(\frac{c_{P2} P_1^2 - c_{P1} P_2^2}{P_1^2 P_2}\right) \tag{10}$$

$$\left(p_P - \frac{c_{P2}}{P_2}\right)\frac{\partial G_2}{\partial P_2} + \left(p_N - \frac{c_{N2}}{N_2}\right)\frac{\partial F_2}{\partial P_2} + \frac{c_{P2}}{P_2^2} G_2 = \left(p_P - \frac{c_{P2}}{P_2}\right)\delta + \sigma_P \left(\frac{c_{P1} P_2^2 - c_{P2} P_1^2}{P_2^2 P_1}\right) \tag{11}$$

This system of implicit equations is a generalization of the optimal selective harvesting equations for a couple biologically interdependent populations (if $\sigma_X = 0$) found by Clark (1976). It is also a generalization of the optimal offshore inshore harvesting equations for a single population found by the same author (if α_i in $F_i(N_i, P_i)$ is zero or if β_i in $G_i(N_i, P_i)$ is zero).

The equation shows that if the costs of harvesting are negligible then the dispersal of individuals does not affect the equlibrium population levels. In this case, numerical examples reveal that the equlibrium population level decreases by the amount of the multiplication of the discounting rate, δ. Figure 1 show the equilibrium population size for prey population with K_1=K_2=400.000 as a function of the predator-prey interaction, α, and the ratio of the price of the predator and the prey, p_P / p_N. It shows that when α is very small, the equilibrium prey size is close to its *maximum sustainable yield* (MSY), K_i/2=200.000. The equilibrium preys size increases significantly with the increasing of α for large values of the ratio p_P / p_N, but it nearly stays the same for small values of the ratio p_P / p_N. It suggests that it is optimal to leave the prey as food for the predator when the price of the predator is higher. Figure 2 shows that if the predator-prey interaction is relatively high but the predator is relatively cheap then the predator should be kept below its maximum sustainable yield to reduce the effect of predatory to the prey (in this example we assume that L_1=L_2=40.000, hence the MSY is L_i/2=20.000). Other simulations can be done to interpret equations (8) - (11) by changing various parameters in the equations.

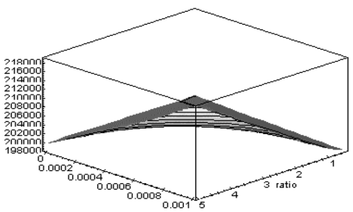

Figure 1

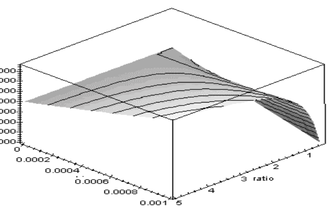

Figure 2

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