A Mathematical Model for Disease Transmission with Age-Structure[1]

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Abstract

Many diseases are known to have specific properties, such as one might be easier to transmit to adults while the other might be easier to infect children. For this reason, incorporating age-structure in modelling the disease transmission will give a better insight into how the disease spreads. However, many mathematical models have ignored this age-structure in analysing the disease transmission. In this paper we discuss a simple mathematical model for disease transmission with the inclusion of age-structure. Some classical questions such as the equilibrium population size, the basic reproductive number and the critical vaccination level are investigated. The results show that some properties to some extent are the generalization of the non-age-structure results while some are quite different from the known non-age-structure results.

Keywords: Disease transmission, Age structure, Minimum vaccination level

1. The Model

Suppose that we divide a population into two groups, adults (group 1) and children (group 2). Among the simplest epidemic model in incorporating this division can be done by modifying the simple SIR model described in [1; p.43] to obtain the following equations:

$$\frac{dS_1}{dt} = B_1 - \beta_{11}S_1I_1 - \beta_{12}S_1I_2 - \mu_1 S_1, \qquad 1$$

$$\frac{dS_2}{dt} = B_2 - \beta_{21}S_2I_1 - \beta_{22}S_2I_2 - \mu_2 S_2, \qquad 2$$

$$\frac{dI_1}{dt} = \beta_{11}S_1I_1 + \beta_{12}S_1I_2 - \mu_1 I_1 - \alpha_1 I_1, \qquad 3$$

$$\frac{dI_2}{dt} = \beta_{21}S_2I_1 + \beta_{22}S_2I_2 - \mu_2 I_2 - \alpha_2 I_2. \qquad 4$$

where S_1 and I_1, respectively, denote the numbers of susceptibles and infectives for the adults, and S_2 and I_2, respectively, denote the numbers of susceptibles and infectives for the children. Let us assume that the birth rate, mortality rate and recovery rate for each group be B_i, μ_i and α_i, respectively. Furthermore, the infection rate in group i is assumed to be proportional to the

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numbers of contact between susceptibles in group i and the infectives from groups i and j. Here β_{ij} denotes the constant of proportionality. In the following sections we investigate the effects of mortality rate μ_i and infection rate β_{ij} to the equilibrium values S_i^* and I_i^*, to the basic reproductive numbers, and to the minimum vaccination levels.

2. The effects of mortality and infection rates to the equilibrium values S_i^* and I_i^*

In this section we discuss the effects of the mortality and infection rates to the values of the equilibriums S_i^* and I_i^*. The effects will be investigated by looking at three different assumptions, namely:

1. Both groups are incorrectly considered to be un-coupled (there is no cross-infection between groups) and β_i are calculated just before the mixing happens (*pre-mixing*)
2. Both groups are incorrectly considered to be un-coupled and β_i are calculated just after the mixing happens (*post-mixing*)
3. Both groups are correctly considered to be coupled (there is cross-infection between groups).

To find the equilibriums of equations (1)-(4) we simplify the notations as follows: $S_{im}^{e*} = S_i$ and $I_{im}^{e*} = I_i$. The LHS of equations (1)-(4) are set to be zero. Next we substitute $\beta_{11}S_1I_1 + \beta_{12}S_1I_2$ and $\beta_{21}S_2I_1 + \beta_{22}S_2I_2$ from equations (3) and (4) into equations (1) and (2) to obtain

$$I_1 = \frac{B_1 - \mu_1 S_1}{(\alpha_1 + \mu_1)} \text{ and } I_2 = \frac{B_2 - \mu_2 S_2}{(\alpha_2 + \mu_2)}. \qquad 5$$

The complete equilibriums are shown in Table 1.

No.	β_i values	(S_1^{e*}, I_1^{e*}) values	(S_2^{e*}, I_2^{e*}) values
1	β_{ii}	$S_1^{e*} = \frac{\mu_1 + \alpha_1}{\beta_{11}}$ $I_1^{e*} = \frac{B_1 - \frac{\mu_1(\mu_1+\alpha_1)}{\beta_{11}}}{\mu_1 + \alpha_1}$	$S_2^{e*} = \frac{\mu_2 + \alpha_2}{\beta_{22}}$ $I_2^{e*} = \frac{B_2 - \frac{\mu_2(\mu_2+\alpha_2)}{\beta_{22}}}{\mu_2 + \alpha_2}$
2	$(\beta_{ii} + \beta_{ij})/2$	$S_1^{e*} = \frac{\mu_1 + \alpha_1}{(\beta_{11}+\beta_{12})/2}$ $I_1^{e*} = \frac{B_1 - \frac{\mu_1(\mu_1+\alpha_1)}{(\beta_{11}+\beta_{12})/2}}{\mu_1 + \alpha_1}$	$S_2^{e*} = \frac{\mu_2 + \alpha_2}{(\beta_{21}+\beta_{22})/2}$ $I_2^{e*} = \frac{B_2 - \frac{\mu_2(\mu_2+\alpha_2)}{(\beta_{21}+\beta_{22})/2}}{\mu_2 + \alpha_2}$
3	β_{ii} dan β_{ij}	$\left(S_{1m}^{e*}, \frac{B_1 - \mu_1 S_{1m}^{e*}}{\mu_1 + \alpha_1}\right)$	$\left(S_{2m}^{e*}, \frac{B_2 - \mu_2 S_{2m}^{e*}}{\mu_2 + \alpha_2}\right)$

Table 1: The equilibrium values for three different assumptions. 1. Separated groups (*pre-mixing*); 2. Separated groups (*post-mixing*); 3. Coupled groups. In the coupled group model, the equilibriums are in implicit form.

To compare the values of the equilibriums we substitute back the resulting values of I_1 and I_2 into equations (1) and (2), which were equating to zero beforehand, to obtain

$$B_1 - \beta_{11}S_1\frac{B_1-\mu_1 S_1}{(\alpha_1+\mu_1)} - \beta_{12}S_1\frac{B_2-\mu_2 S_2}{(\alpha_2+\mu_2)} - \mu_1 S_1 = 0, \tag{6}$$

$$B_2 - \beta_{21}S_2\frac{B_1-\mu_1 S_1}{(\alpha_1+\mu_1)} - \beta_{22}S_2\frac{B_2-\mu_2 S_2}{(\alpha_2+\mu_2)} - \mu_2 S_2 = 0. \tag{7}$$

The last two equations were solved (in this case we use Maple V), by considering $\alpha_1=\alpha_2=\alpha$ and $B_1=B_2=B$, to find an implicit form

$$S_2 = \frac{X_2+Y_2}{\beta_{12}\mu_2(\alpha+\mu_1)S_1}, \tag{8}$$

In which

$$X_2 = -\left(B\alpha^2 + B\alpha\mu_2 + B\alpha\mu_1 + B\mu_1\mu_2 - \beta_{11}B\alpha S_1 - \beta_{11}B\mu_2 S_1 - \beta_{12}B\alpha S_1 - \beta_{12}B\mu_1 S_1\right)$$

and

$$Y_2 \;=\; -\left(-\mu_1\alpha^2 S_1 - \alpha\mu_1\mu_2 S_1 - \alpha\mu_1{}^2 S_1 - \mu_1{}^2\mu_2 S_1 + \beta_{11}\alpha\mu_1 S_1{}^2 + \beta_{11}\mu_1\mu_2 S_1{}^2\right).$$

The expression of X_2 can be simplified into

$$X_2 \;=\; -B\left(\alpha(\alpha+\mu_2) + \mu_1(\alpha+\mu_2) - S_1(\beta_{11}\alpha + \beta_{11}\mu_2 + \beta_{12}\alpha + \beta_{12}\mu_1)\right).$$

The next simplifications of X_2 and Y_2 yields

$$X_2 \;=\; -B\left((\alpha+\mu_1)(\alpha+\mu_2) - S_1(\beta_{11}(\alpha+\mu_2) + \beta_{12}(\alpha+\mu_1)\right), \tag{9}$$

$$Y_2 \;=\; -S_1\left(-(\alpha+\mu_1)\mu_1(\alpha+\mu_2) + \beta_{11}N_1\mu_1(\alpha+\mu_2)\right). \tag{10}$$

Furthermore, to compare the values of equilibrium of the different groups we let $\Delta S = S_1 - S_2$, so $\Delta S = S_1 - \dfrac{X_2+Y_2}{\beta_{12}\mu_2(\alpha+\mu_1)S_1}$. The RHS can be written as

$$= \;\; S_1\beta_{12}\mu_2(\alpha+\mu_1)S_1 + B\left((\alpha+\mu_1)(\alpha+\mu_2) - S_1(\beta_{11}(\alpha+\mu_2) + \beta_{12}(\alpha+\mu_1)\right)$$
$$+ S_1\left(-(\alpha+\mu_1)\mu_1(\alpha+\mu_2) + \beta_{11}S_1\mu_1(\alpha+\mu_2)\right)$$
$$= S_1\beta_{12}(\alpha+\mu_1)(\mu_2 S_1 - B) + (B-\mu_1 S_1)(\alpha+\mu_1)(\alpha+\mu_2) + S_1\beta_{11}(\alpha+\mu_2)(\mu_1 S_1 - B)$$
$$= (B-\mu_1 S_1)\left[(\alpha+\mu_1)(\alpha+\mu_2) - S_1\beta_{11}(\alpha+\mu_2)\right] + S_1\beta_{12}(\alpha+\mu_1)(\mu_2 S_1 - B)$$
$$= (B-\mu_1 S_1)\left[(\alpha+\mu_2)\left((\alpha+\mu_1) - S_1\beta_{11}\right)\right] + S_1\beta_{12}(\alpha+\mu_1)(\mu_2 S_1 - B).$$

It is easy to show that if $\mu_1=\mu_2=\mu$, $\beta_{12}=\beta_{21}=0$ and $\beta_{11}=\beta_{22}=\beta$ then equilibrium values for both groups are equal. The discussion in [2] shows that if $\mu_1 \neq \mu_2$ and there are no cross infection between the two group, i.e. $\beta_{12}=\beta_{21}=0$, then the equilibriums for both groups can be obtained from a single (unstructured) model.

Next we see the following case of $\mu_1 \neq \mu_2$ and $\beta_{11}=\beta_{12}=\beta_{21}=\beta_{22}=\beta\neq 0$. In this case we assume that the mortality of the two groups are different and each group can infect

each other. We will show that $S_1 \neq S_2$. We will also show that conclusion if we incorrectly assume that there is no cross-infection between groups will significantly different from the case if we correctly assume that indeed there is a cross-infection between the groups.

Note that from $\Delta S = S_1 - S_2$ we can derive

$$S_1 - S_2 = (B-\mu_1 S_1)\left[(\alpha+\mu_2)\left((\alpha+\mu_1) - S_1\beta_{11}\right)\right] + S_1\beta_{12}(\alpha+\mu_1)(\mu_2 S_1 - B) > 0$$

provided $(B-\mu_1 S_1)=0$ and $(\mu_2 S_1 - B) > 0$. Alternatively $B/S_1 = \mu_1$ and $B/S_1 < \mu_2$, means that $\mu_1 < \mu_2$. This concludes that $\mu_1 < \mu_2 \Rightarrow S_1 > S_2$. This result contradicts the result if we incorrectly assume that there is no cross-infection between the groups, that is $S_1^{e*} = \dfrac{\mu_1+\alpha}{\beta} < \dfrac{\mu_2+\alpha}{\beta} = S_2^{e*}$. We give a numerical example to illustrate this phenomenon in [2].

3. The minimum vaccination level

Let us assume that a portion p_i of susceptible population is vaccinated so that the dynamics is given by equations (1) to (4) except

$$\frac{dS_i}{dt} = (1-p_i)B_i - \beta_{ii}S_i I_i - \beta_{ij}S_i I_j - \mu_i S_i. \tag{11}$$

Equilibrium values of the system, now are given by

$$I_{vi}^* = \frac{(1-p_i)B_i - \mu_i S_{vi}^*}{\mu_i+\alpha_i} = \mu_i S_{vi}^*\frac{\dfrac{(1-p_i)B_i}{\mu_i S_{vi}^*} - 1}{\mu_i+\alpha_i} = \mu_i S_{vi}^*\frac{(1-p_i)R_{0i} - 1}{\mu_i+\alpha_i}, \tag{12}$$

with $R_{0i} = \dfrac{B_i/\mu_i}{S_{vi}^*} = \dfrac{S_{0i}}{S_{vi}^*}$. It is clear that eventually there will be no infective if $(1-p_i)R_{0i} \le 1$ means that $p_i \ge 1 - \dfrac{1}{R_{0i}}$. We conclude that the *critical vaccination level* for each group is given by $p_{ci} = 1 - 1/R_{0i}$. This value has the same expression as the critical vaccination level for many known structured or un-structured population models [3].

4. The computation of the basic reproduction ratio

In this section we derive the *basic reproductive ratio*, R_0, for the system in equations (1) to (4). In this system there are two types of infections: type 1 infected by group 1 and type 2 infected by group 2. To find the R_0, we construct the next generation matrix $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ where k_{ij} = expected number of new cases of type *i* infected by one case of type *j*. Hence, $k_{ij} = \beta_{ij}\overline{S_i}\left(\dfrac{1}{\mu_j+\alpha_j}\right)$ with $\overline{S_i}$ denotes the disease free susceptible equilibrium, *i.e* $\overline{S_i} = \dfrac{B_i}{\mu_i}$.

The full matrix is given by

$$K = \begin{pmatrix} \frac{\beta_{11}\overline{S_1}}{\alpha_1+\mu_1} & \frac{\beta_{12}\overline{S_1}}{\alpha_2+\mu_2} \\ \frac{\beta_{21}\overline{S_2}}{\alpha_1+\mu_1} & \frac{\beta_{22}\overline{S_2}}{\alpha_2+\mu_2} \end{pmatrix}. \quad 13$$

In [1] it is found that R_0 is the dominant eigen value of the matrix K. Since the eigen value for 2x2 matrix is given by the root of the characteristic equation $\lambda^2 - Trace(K)\lambda + Det(K) = 0$, it can be proved that if $\beta_{ij} = a_i b_i$ then

$$R_0 = \frac{1}{2}\left(\frac{\beta_{11}\overline{S_1}}{\alpha_1+\mu_1} + \frac{\beta_{22}\overline{S_2}}{\alpha_2+\mu_2}\right). \quad 14$$

In this case R_0 for the coupled groups is the average of the R_0 for the un-coupled groups. The knowledge of R_0 both for coupled or uncoupled groups, and its changes to the change of other parameters, is very important to compare various decisions in controlling the disease. Relevant question regarding an age-structure, such as which group should have a high priority to be vaccinated can be address further. We examine the effects of the mortality and infection rates to the values of R_0, and hence to the critical vaccination level, by investigating various cases of β_{ij}, elsewhere.

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