A PORTFOLIO SELECTION PROBLEM WITH POSSIBILISTIC APPROACH

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Abstract

We consider a mathematical programming model with probabilistic constraints and we solve it by transforming this problem into a multiple objective linear programming problem. Also we obtain some results by using the approach of crisp weighted possibilistic mean value of fuzzy number.

Keyword : Portfolio selection, VaR, Possibilistic mean value, Possibilistic theory, Weighted possibilistic mean, Fuzzy theory.

AMS subject classifications. Primary, 90B28, 90C15, 90C29, 90C48, 90C70; Secondary, 46N10, 60E15, 91B06, 90B10

1. INTRODUCTION

Predictions about investor portfolio holdings can provide powerful tests of asset pricing theories. In the context of Markowitz portfolio selection problem, this paper develops a possibilistic mean VaR model with multi assets. Furthermore, through the introduction of a set of investor-specific characteristics, the methodology accommodates either homogeneous or heterogeneous anticipated rates of return models. Thus we consider a mathematical programming model with probabilistic constraint and we it solve by transforming this problem into a multiple objective linear programming problem. Also we obtain our results by using the approach of crisp weighted possibilistic mean value risk and possibilistic mean variance of fuzzy number.

The rest of the paper is organized in the following manner. Section 2, propose a formulation of mean VaR the portfolio selection model with multi assets problem. Section 3, consider an overview of the possibility theory and propose a possibilistic mean VaR portfolio selection model. Also some results relatively to efficient portfolios are stated. In section 4, are obtained some results for efficient portfolios in the frame of the weighted possibilistic mean value approach. Thus are extended some recently results in this field [4, 6, 7].

2. MEAN *VaR* PORTFOLIO SELECTION MULTIOBJECTIVE MODEL WITH TRANSACTION COSTS

We begin by using the rates of return of the risky securities in the economic when we have a multivariate normal distribution. In practice, the computing of a portfolio's *VaR* this is a popular assumption when (see Hull and White [11]).

2.1 MEAN DOWNSIDE-RISK FRAMEWORK

In this section we extended [7, 8, 12] for i assets. In practice investors are concerned about the risk that their portfolio value falls below a certain level. That is the reason why different measures of downside-risk are considered in the multi asset allocation problems. Denoted the random bariable $\upsilon_i, i = \overline{1,q}$ the future portfolio value, i.e., the value of the portfolio by the end of the planning period, then the probability

$P(\upsilon_i < (VaR)_i)$, $i = \overline{1,q}$,

that υ_i the portfolio value falls below the $(VaR)_i$ level, is called the shortfall probability. The conditional mean value of the portfolio given that the portfolio value has fallen below $(VaR)_i$, called the expected shortfall, is defined as

$E(\upsilon_i \,|\upsilon_i < (VaR)_i)$.

Other risk measures used in practice are the mean absolute deviation

$E\left\{(|\upsilon_i - E(\nu_i)|)\,|\upsilon_i < E(\upsilon_i)\right\}$,

and the semi-variance

$E((\upsilon_i - E(\nu_i))^2\,|\upsilon_i < E(\upsilon_i))$

where we consider only the negative deviations from the mean.

Let $x_j (j = \overline{1,n})$ represents the proportion of the total amount of money devoted to security j and M_{1j} and M_{2j} represent the minimum and maximum proportion of the total amount of money devoted to security j, respectively. For $j = \overline{1,n}, i = \overline{1,q}$, let r_{ji} be a random variable which is the rate of the i return of security j. Then $\upsilon_i = \sum\limits_{j=1}^{n} r_{ji}x_j$.

Assume that an investor wants to allocate his/her wealth among n risky securities. If the risk profile of the investor is determined in terms of $(VaR)_i, i = \overline{1,q}$, a mean-$VaR$ efficient portfolio will be a solution of the following multiobjective optimization problem

$$Max\ [E(\upsilon_1), ..., E(\upsilon_k)] \tag{2.1}$$

$$subject\ to\ P_r\{\upsilon_i \leq (VaR)_i\} \leq \beta_i, i = 1, ..., k, \tag{2.2}$$

$$\sum_{j=1}^{n} x_j = 1 \tag{2.3}$$

$$M_{1j} \leq x_j \leq M_{2j},\ j = i = \overline{1,n}. \tag{2.4}$$

In this model, the investor is trying to maximize the future value of his/her portfolio, which requires the probability that the future value of his portfolio falls below $(VaR)_i$ not to be greater than $\beta_i, i = \overline{1,q}$.

2.2. THE PROPORTIONAL TRANSACTION COST MODEL

The introduction of transaction costs adds considerable complexity to the optimal portfolio selection problem. The problem is simplified if one assumes that the transaction costs are proportional to the amount of the risky asset traded, and there are no transaction costs on trades in the riskless asset. Transaction cost is one of the main sources of concern to managers [1, 17.

Assume the rate of transaction cost of security $j(j=\overline{1,n})$ and allocation of $i(i=\overline{1,q})$ asset is c_{ji}.Thus the transaction cost of security j and allocation of i assets is $c_{ji}x_j$. The transaction cost of portfolio $x=(x_1,...,x_q)$ is $\sum\limits_{j=1}^{n} c_{ji}x_j, i=\overline{1,q}$. Considering the proportional transaction cost and the shortfall probability constraint, we purpose the following mean VaR portfolio selection model with transaction costs:

$$(2.5) \qquad Max\left[E(v_1)-\sum_{j=1}^{n} c_{j1}x_j-...-E(v_k)-\sum_{j=1}^{n} c_{jk}x_j\right]$$

$$(2.6) \qquad subject\ to\ P_r\{v_i\leq (VaR)_i\}\leq \beta_i, i=\overline{1,q},$$

$$(2.7) \qquad \sum_{j=1}^{n} x_j=1,$$

$$(2.8) \qquad M_{1j}\leq x_j\leq M_{2j},\ j=\overline{1,n}.$$

3. POSSIBILISTIC MEAN VaR PORTFOLIO SELECTION MODEL

3.1 POSSIBILISTIC THEORY

We consider the possibilistic theory proposed by Zadeh [17]. Let $\widetilde{a}$ and $\widetilde{b}$ be two fuzzy numbers with membership functions $\mu_{\widetilde{a}}$ and $\mu_{\widetilde{b}}$ respectively. The possibility operator (Pos) is defined as follows [5].

$$(3.1) \qquad \begin{cases} Pos(\widetilde{a}\leq\widetilde{b})=\sup\{\min(\mu_{\widetilde{a}}(x),\mu_{\widetilde{b}}(y))\,|x,y\in\mathbb{R},x\leq y\} \\ Pos(\widetilde{a}<\widetilde{b})=\sup\{\min(\mu_{\widetilde{a}}(x),\mu_{\widetilde{b}}(y))\,|x,y\in\mathbb{R},x< y\} \\ Pos(\widetilde{a}=\widetilde{b})=\sup\{\min(\mu_{\widetilde{a}}(x),\mu_{\widetilde{b}}(x))\,|x\in\mathbb{R}\} \end{cases}$$

In particular, when $\widetilde{b}$ is a fuzzy number b, we have

$$(3.2) \qquad \begin{cases} Pos(\widetilde{a}\leq b)=\sup\{\mu_{\widetilde{a}}(x)\,|x\in\mathbb{R},x\leq b\} \\ Pos(\widetilde{a}< b)=\sup\{\mu_{\widetilde{a}}(x)\,|x\in\mathbb{R},x< b\} \\ Pos(\widetilde{a}=b)=\mu_{\widetilde{a}}(b). \end{cases}$$

Let $f:\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ be a binary operation over real numbers. Then it can be extended to the operation over the set of fuzzy numbers. If we denoted for the fuzzy numbers $\widetilde{a},\widetilde{b}$ the numbers $\widetilde{c}=f(\widetilde{a},\widetilde{b})$, then the membership function $\mu_{\widetilde{c}}$ is obtained from the membership function $\mu_{\widetilde{a}}$ and $\mu_{\widetilde{b}}$ by

$$(3.3) \qquad \mu_{\widetilde{c}}(z)=\sup\{\min(\mu_{\widetilde{a}}(x),\mu_{\widetilde{b}}(y))\,|x,y\in\mathbb{R},z=f(x,y)\}$$

for $z\in\mathbb{R}$. That is, the possibility that the fuzzy number $\widetilde{c}=f(\widetilde{a},\widetilde{b})$ achives value $z\in\mathbb{R}$ is as great as the most possibility combination of real numbers x,y such that $z=f(x,y)$, where the value of $\widetilde{a}$ and $\widetilde{b}$ are x and y respectively.

3.2 TRIANGULAR AND TRAPEZOIDAL FUZZY NUMBERS

Let the rate of return on security given by a trapezoidal fuzzy number $\widetilde{r}=(r_1,r_2,r_3,r_4)$ where $r_1<r_2\leq r_3<r_4$, and the membership function of the fuzzy number $\widetilde{r}$ can be denoted by:

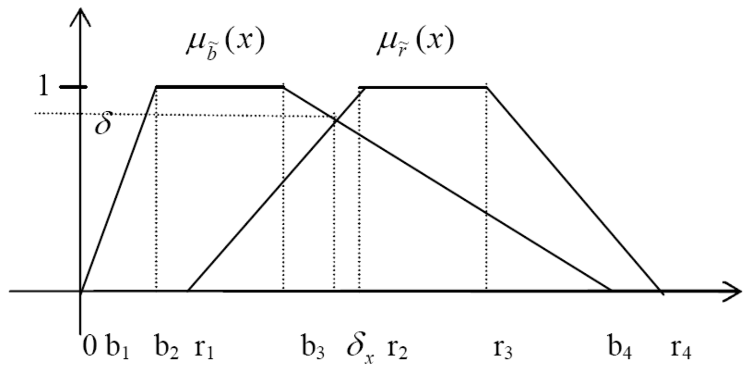

Figura 3.1: Two Trapezoidal Fuzzy Number $\widetilde{r}$ and $\widetilde{b}$.

$$\mu_{\widetilde{r}}(x) = \begin{cases} \frac{x-r_1}{r_2-r_1} & , r_1 \leq x \leq r_2 \\ 1 & , r_2 \leq x \leq r_3 \\ \frac{x-r_4}{r_3-r_4} & , r_3 \leq x \leq r_4 \\ 0 & , otherwise. \end{cases} \quad (3.4)$$

We mention that trapezoidal fuzzy number is triangular fuzzy number if $r_2 = r_3$.

Let us consider two trapezoidal fuzzy numbers $\widetilde{r} = (r_1, r_2, r_3, r_4)$ and $\widetilde{b} = (b_1, b_2, b_3, b_4)$, as shown in Figure 3.1.

If $r_2 \geq b_3$, then we have

$$Pos(\widetilde{r} \leq \widetilde{b}) = \sup\{\min(\mu_{\widetilde{r}}(x), \mu_{\widetilde{b}}(y)) \,|x \leq y\}$$
$$\geq \min\{(\mu_{\widetilde{r}}(r_2), \mu_{\widetilde{b}}(b_3)\} = \min\{1, 1\} = 1,$$

which implies that $Pos(\widetilde{r} \leq \widetilde{b}) = 1$. If $r_2 \geq b_3$ and $r_1 \leq b_4$ then the supremum is achieved at point of intersection δ_x of the two membership function $\mu_{\widetilde{r}}(x)$ and $\mu_{\widetilde{b}}(x)$. A simple computation shows that

$$Pos\{\widetilde{r} \leq \widetilde{b}\} = \delta = \frac{b_4 - r_1}{(b_4 - b_3) + (r_2 - r_1)}$$

and

$$\delta_x = r_1 + (r_2 - r_{1)}\delta$$

If $r_1 > b_4$, then for any $x < y$, at least one of the equalities

$$\mu_{\widetilde{r}}(x) = 0, \mu_{\widetilde{b}}(y) = 0$$

hold. Thus we have $Pos\{\widetilde{r} \leq \widetilde{b}\} = 0$. Now we summarize the above results as

$$Pos\{\widetilde{r} \leq \widetilde{b}\} = \begin{cases} 1 & , r_2 \leq b_3 \\ \delta & , r_2 \geq b_3, r_1 \leq b_4 \\ 0 & , r_1 \geq b_4. \end{cases} \quad (3.5)$$

Especially, when $\widetilde{b}$ is the crisp number 0, then we have

(3.6) $$Pos\{\widetilde{r}\leq 0\}=\begin{cases}1 & ,r_2\leq 0\\ \delta & ,r_1\leq 0\leq r_2\\ 0 & ,r_1\geq 0\end{cases}$$

where

(3.7) $$\delta=\frac{r_1}{r_1-r_2}.$$

We now turn our attention the following lemma.

LEMMA 3.1 [5] Let $\widetilde{r}=(r_1,r_2,r_3,r_4)$ be *trapezoidal fuzzy number. Then for any given confidence level* α *with* $0\leq\alpha\leq 1, Pos(\widetilde{r}\leq 0)\geq\alpha$ *if and only if* $(1-\alpha)r_1+\alpha r_2\leq 0.$

The λ *level set of a fuzzy number* $\widetilde{r}=(r_1,r_2,r_3,r_4)$ *is a crisp subset of* $\mathbb{R}$ *and denoted by* $[\widetilde{r}]^{\lambda}=\{x\,|\mu(x)\geq\lambda, x\in\mathbb{R}\}$ *, then according to Carlsson et al* [4] *we have*

$$[\widetilde{r}]^{\lambda}=\{x\,|\mu(x)\geq\lambda, x\in\mathbb{R}\}=[r_1+\lambda(r_2-r_1),r_4-\lambda(r_4-r_3)].$$

Given $[\widetilde{r}]^{\lambda}=\{a_1(\lambda),a_2(\lambda)\}$, *the crisp possibilistic mean value* of $\widetilde{r}=(r_1,r_2,r_3,r_4)$ is

$$\widetilde{E}(\widetilde{r})=\int_0^1\lambda(a_1(\lambda)+a_2(\lambda)d\lambda.$$

where $\widetilde{E}$ *denotes fuzzy mean operator.*

We can see that if $\widetilde{r}=(r_1,r_2,r_3,r_4)$ is a trapezoidal fuzzy number then

(3.8) $$\widetilde{E}(\widetilde{r})=\int_0^1\lambda(r_1+\lambda(r_2-r_1)+r_4-\lambda(r_4-r_3))d\lambda=\frac{r_2+r_3}{3}+\frac{r_1+r_4}{6}$$

3.3. EFFICIENT PORTFOLIOS

Let x_j the proportional of the total amount of money devoted to security j , M_{1_j} and M_{2_j} represent the minimum and maximum proportion respectively of the total amount of money devoted to security j . The trapezoidal fuzzy number of r_{ji} is $\widetilde{r}_{ji}=(r_{(ji)1},r_{(ji)2},r_{(ji)3},r_{(ji)4})$ where $r_{(ji)1}<r_{(ji)2}\leq r_{(ji)3}<r_{(ji)4}$. In addition, we denote the $(VaR)_i$ level by the fuzzy number trapezoidal $\widetilde{b}=(b_{1i},b_{i2},b_{i3},b_{i4}), i=1,...,k.$

Using this approach we see that the model given by (2.5)-(2.8) $E(v_i)$ reduces to the form from the following theorem..

THEOREM 3.1 *The possibilistic mean* VaR *portfolio model for the vector mean* VaR *efficient portfolio mode*l (2.5)-(2.8) is

(3.9) $$\max\left\{\widetilde{E}\left(\sum_{i=1}^{n}\widetilde{r}_{j1}x_j\right)-\sum_{i=1}^{n}c_{j1}x_j-...-\widetilde{E}\left(\sum_{i=1}^{n}\widetilde{r}_{jk}x_j\right)-\sum_{i=1}^{n}c_{jk}x_j\right.$$

(3.10) $$\text{s.t } Pos\left(\sum_{i=1}^{n}\widetilde{r}_{jk}x_j<\widetilde{b}i\right)\leq\beta_i, i=\overline{1,q},$$

(3.11) $$\sum_{i=1}^{n}x_j=1,$$

(3.12) $$M_{1j}\leq x_j\leq M_{2j}, \text{jj}=\overline{1,n}.$$

In the following for obtain the efficient portfolios given by Theorem 3.1 we use White [15].

THEOREM 3.2 If $\lambda_i > 0, i = 1, ..., k$, *then an efficient portfolio for possibilistic model is an optimal solution of the following problem* :

$$(3.13) \qquad \max \sum_{i=1}^{n} \lambda_i \left[\widetilde{E} \left(\sum_{i=1}^{n} \widetilde{r}_{ji} x_j \right) - \sum_{i=1}^{n} c_{ji} x_j \right]$$

$$(3.14) \qquad \text{s.t } Pos \left(\sum_{i=1}^{n} \widetilde{r}_{jk} x_j < \widetilde{b}i \right) \leq \beta_i, i = \overline{1,q},$$

$$(3.15) \qquad \sum_{i=1}^{n} x_j = 1,$$

$$(3.16) \qquad M_{1j} \leq x_j \leq M_{2j}, \; j = \overline{1,n}.$$

Using the fact that rate of return on security $j (j = \overline{1,n})$ by trapezoidal fuzzy number, from which the required result follows.

THEOREM 3.3. *Let rate of return on security* $j (j = \overline{1,n})$ *by the trapezoidal* $\widetilde{r}_{ji} = (r_{(ji)1}, r_{(ji)2}, r_{(ji)3}, r_{(ji)4})$ where $r_{(ji)1} < r_{(ji)2} \leq r_{(ji)3} < r_{(ji)4}$. *In addition* $\widetilde{b} = (b_{1i}, b_{i2}, b_{i3}, b_{i4})$ *is trapezoidal fuzzy number for* VaR *level and* $\lambda > 0$, *with* $i = \overline{1,q}$*.Then using the possibilistic mean* VaR *portfolio selection model on efficient portfolio is an optimal solution for the following problem* :

$$(3.17) \qquad \max \sum_{i=1}^{n} \lambda_i \left[\frac{\sum_{i=1}^{n} r_{(ji)2} x_j + \sum_{i=1}^{n} r_{(ji)3} x_j +}{3} + \frac{\sum_{i=1}^{n} r_{(ji)1} x_j + \sum_{i=1}^{n} r_{(ji)4} x_j}{6} - \sum_{i=1}^{n} c_{ji} x_j \right]$$

$$(3.18) \qquad \text{s.t } (1 - \beta_i) \left(\sum_{i=1}^{n} r_{(ji)1} x_j - bi4 \right) + \beta_i \left(\sum_{j=1}^{n} r_{(ji)2} x_j - b_{i3} \right) \geq 0, i = \overline{1,q},$$

$$(3.19) \qquad \sum_{i=1}^{n} x_j = 1,$$

$$(3.20) \qquad M_{1j} \leq x_j \leq M_{2j}, \; j = \overline{1,n}.$$

Proof : Really, from the equation (3.8), we have

$$\widetilde{E} \left(\sum_{i=1}^{n} \widetilde{r}_{ji} x_j \right) = \left[\frac{\sum_{i=1}^{n} r_{(ji)2} x_j + \sum_{i=1}^{n} r_{(ji)3} x_j}{3} + \frac{\sum_{i=1}^{n} r_{(ji)1} x_j + \sum_{i=1}^{n} r_{(ji)4} x_j +}{6} - \sum_{i=1}^{n} c_{ji} x_j \right],$$

.

From Lemma 3.1, we have that

$Pos \left(\sum_{i=1}^{n} \widetilde{r}_{jk} x_j < \widetilde{b}i \right) \leq \beta_i, i = \overline{1,q}$, is equivalent with

$$(1 - \beta_i) \left(\sum_{j=1}^{n} r_{(ji)1} x_j - b_{i4} \right) + \beta_i \left(\sum_{j=1}^{n} r_{(ji)2} x_j - b_{i3} \right) \geq 0.$$

Furthermore, from (3.17)-(3.20) given by Theorem 3.2, we get is the following form :

$$(3.21) \qquad \max_{x \in \mathbb{R}^n} \sum_{i=1}^{k} \lambda_i \left[\frac{\sum_{j=1}^{n} r_{(ji)2} x_j + \sum_{j=1}^{n} r_{(ji)3} x_j}{3} + \frac{\sum_{j=1}^{n} r_{(ji)1} x_j + \sum_{j=1}^{n} r_{(ji)4} x_j}{6} - \sum_{j=1}^{n} c_{ji} x_j \right]$$

$$(3.22) \qquad \text{s.t. } \; (1 - \beta_i) \left(\sum_{j=1}^{n} r_{(ji)1} x_j - b_{i4} \right) + \beta_i \left(\sum_{j=1}^{n} r_{(ji)2} x_j - b_{i3} \right) \geq 0, i = \overline{1,q},$$

(3.23) $\sum\limits_{i=1}^{n} x_j = 1,$

(3.24) $M_{1j} \leq x_j \leq M_{2j},\ j = i = \overline{1,n}.$

This completes the proof. $\blacksquare$

Problem (3.21)-(3.24) is a standard multi-objective linear programming problem. For optimal solution we can used several algorithm of multi-objective programming [12, 14].

4. WEIGHTED POSSIBILISTIC MEAN VALUE APPROACH

In this section introducing a weighting function measuring the importance of λ-level sets of fuzzy numbers we consider define the weighted lower possibilistic and upper possibilistic mean values, crisp possibilistic mean value of fuzzy numbers, which is consistent with the extension principle and with the well-known definitions of expectation in probability theory. We shall also show that the weighted interval-valued possibilistic mean is always a subset (moreover a proper subset excluding some special cases) of the interval-valued probabilistic mean for any weighting function.

A trapezoidal fuzzy number $\widetilde{r} = (r_1, r_2, r_3, r_4)$ $\blacksquare$is a fuzzy set of the real line $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by $\mathcal{F}$. A λ-level set of a fuzzy number $\widetilde{r} = (r_1, r_2, r_3, r_4)$ $\blacksquare$is defined by $[\widetilde{r}]^{\lambda} = \{x \,|\mu(x) \geq \lambda,\ x \in \mathbb{R}\}$, then$\blacksquare$

$[\widetilde{r}]^{\lambda} = \{x \,|\mu(x) \geq \lambda,\ x \in \mathbb{R}\} = [r_1 + \lambda(r_2 - r_1), r_4 - \lambda(r_1 - r_4)],$

if $\lambda > 0$ and $[\widetilde{r}]^{\lambda} = cl\{x \in \mathbb{R} \,|\mu(x) \geq 0\ \}$ (the closure of the support of $\widetilde{r}$) if $\lambda = 0$. It is well-known that if $\widetilde{r}$ $\blacksquare$is a fuzzy number then $[\widetilde{r}]^{\lambda}$ $\blacksquare$is a compact subset of $\mathbb{R}$ for all $\lambda[0, 1]$.

DEFINITION 4.1 [6] *Let $\widetilde{r} \in \mathcal{F}$ $\blacksquare$ be fuzzy number with* $[\widetilde{r}]^{\lambda} = [a_1(\lambda), a_2(\lambda)], \lambda \in [0, 1]$. *A function $w : [0, 1] \to \mathbb{R}$ is said to be a weighting function if w is non-negative, monoton increasing and satisfies the following normalization condition*

(4.1) $\int\limits_0^1 w(\lambda)d\lambda = 1.$

The w-weighted possibilistic mean (or expected) value of fuzzy number $\widetilde{r}$ is

(4.2) $\overline{\overline{E}}_w(\widetilde{r}) = \int\limits_0^1 \frac{a_1(\lambda)+a_2(\lambda)}{3} w(\lambda)d\lambda.$

It should be noted that if then

$$\overline{\overline{E}}_w(\widetilde{r}) = \int\limits_0^1 [a_1(\lambda) + a_2(\lambda)]\lambda d\lambda.$$

That is the w-weighted possibilistic mean value defined by (4.2) can be considered as a generalization of possibilistic mean value in [6]. From the definition of a weighting function it can be seen that $w(\lambda)$ might be zero for certain (unimportant) λ-level sets of $\widetilde{r}$. So by introducing different weighting functions we can give different (case-dependent) important to λ-levels sets of fuzzy numbers.

Let $\widetilde{r} = (r_1, r_2, \alpha, \beta)$ $\blacksquare$be a fuzzy number of trapezoidal form and with peak $[r_1, r_2]$, left-width $\alpha > 0$ and right-width $\beta > 0$ and let

$w(\lambda) = (2q-1)[(1-\lambda)^{-1/2q} - 1]$,

where $q > 1$. It's clear that w is weighting function with $w(0) = 0$ and $\lim\limits_{q \to 1-0} w(\lambda) = \infty$.

Then the w-weighted lower and upper possibilistic mean values of $\widetilde{r}$ are computed by

$$\widetilde{E}_w^-(\widetilde{r}) = \int_0^1 [r_1 - (1-\lambda)\alpha] 2q[(1-\lambda)^{-1/2q} - 1] d\lambda$$
$$= r_1 - \tfrac{q\alpha}{4q-1}.$$

and

$$\widetilde{E}_w^+(\widetilde{r})(\widetilde{r}) = \int_0^1 [r_2 + (1-\lambda)\beta] 2q[(1-\lambda)^{-1/2q} - 1] d\lambda$$
$$= r_2 + \tfrac{q\beta}{4q-1}$$

and therefore

$$\widetilde{E}_w(\widetilde{r}) = [r_1 - \tfrac{q\alpha}{4q-1}, r_2 + \tfrac{q\beta}{4q-1}]$$

$$\overline{\widetilde{E}}_w(\widetilde{r}) = \tfrac{r_1+r_2}{2} + \tfrac{q(\beta-\alpha)}{2(4q-1)}. \tag{4.3}$$

This observation along with Theorem 3.1 as section 3.3 leads to the following theorem.

THEOREM 4.1 *The mean VaR efficient portfolio model is*

$$\max_{x \in \mathbb{R}^n} \sum_{i=1}^{k} \lambda_i \left[\overline{\widetilde{E}}_w \left(\sum_{j=1}^{n} \widetilde{r}_{ji} x_j \right) - \sum_{i=1}^{k} c_{ji} x_j \right] \tag{4.4}$$

$$\text{s.t. } Pos\left(\sum_{j=1}^{n} \widetilde{r}_{ji} x_j < \widetilde{b}_i \right) \leq \beta_i, i = 1, ..., k, \tag{4.5}$$

$$\sum_{i=1}^{n} x_j = 1, \tag{4.6}$$

$$M_{1j} \leq x_j \leq M_{2j},\ j = 1, ..., n. \tag{4.7}$$

In the next theorem we extend Theorem 3.3 to the case weighted possibility mean value approach $w(\lambda)$.

THEOREM 4.2. *Let $w(\lambda) = 2q[(1-\lambda)^{-\frac{1}{2q}} - 1], q > 1$ is weighted possibility mean of fuzzy number $\widetilde{r}_{ji}$ and let rate of return on security $j (j = 1, ..., n)$ by the trapezoidal number $\widetilde{r}_{ji} = (r_{(ji)1}, r_{(ji)2}, r_{(ji)3}, r_{(ji)4})$ where $r_{(ji)1} < r_{(ji)2} \leq r_{(ji)3} < r_{(ji)4}$ and addition $\widetilde{b} = (b_{1i}, b_{i2}, b_{i3}, b_{i4})$* is *trapezoidal number for $(VaR)_i, i = 1, ..., k$. Then the possibilistic mean VaR portfolio selection model is*

$$\max_{x \in \mathbb{R}^n} \sum_{i=1}^{k} \lambda_i \left[\frac{\sum\limits_{j=1}^{n} r_{(ji)1} x_j + \sum\limits_{j=1}^{n} r_{(ji)2} x_j}{2} + \frac{q \left(\sum\limits_{j=1}^{n} r_{(ji)1} x_j - \sum\limits_{j=1}^{n} r_{(ji)4} x_j \right)}{2(4q-1)} - \sum_{j=1}^{n} c_{ji} x_j \right] \tag{4.8}$$

$$s.t. \quad (1-\beta_i) \left(\sum_{j=1}^{n} r_{(ji)1} x_j - b_{i4} \right) + \beta_i \left(\sum_{j=1}^{n} r_{(ji)2} x_j - b_{i3} \right) \geq 0, i = \overline{1, q}, \tag{4.9}$$

$$\sum_{j=1}^{n} x_j = 1, \tag{4.10}$$

$$M_{1j} \leq x_j \leq M_{2j},\ j = \overline{1, n}. \tag{4.11}$$

Proof : Really, from the equation (4.2), we have

$$\overline{\widetilde{E}}\left(\sum_{j=1}^{n}\widetilde{r}_{ji}x_j\right)=\frac{\sum_{j=1}^{n}r_{(ji)1}x_j+\sum_{j=1}^{n}r_{(ji)2}x_j+}{2}+\frac{q\left(\sum_{j=1}^{n}r_{(ji)1}x_j-\sum_{j=1}^{n}r_{(ji)4}x_j\right)}{2(4q-1)},i=\overline{1,q}.$$

From Lemma 3.1, we have that

$Pos\left(\sum_{i=1}^{n}\widetilde{r}_{jk}x_j<\widetilde{b}i\right)\leq\beta_i, i=1,...,k$, is equivalent with

$$(1-\beta_i)\left(\sum_{j=1}^{n}r_{(ji)1}x_j-b_{i4}\right)+\beta_i\left(\sum_{j=1}^{n}r_{(ji)2}x_j-b_{i3}\right)\geq 0.$$

.

Furthermore, from (4.8)-(4.11) given by Theorem 4.1, is the following form :

$$\max_{x\in\mathbb{R}^n}\sum_{i=1}^{k}\lambda_i\left[\frac{\sum_{j=1}^{n}r_{(ji)1}x_j+\sum_{j=1}^{n}r_{(ji)2}x_j}{2}+\frac{q\left(\sum_{j=1}^{n}r_{(ji)1}x_j-\sum_{j=1}^{n}r_{(ji)4}x_j\right)}{2(4q-1)}-\sum_{j=1}^{n}c_{ji}x_j\right] \tag{4.12}$$

$$\text{s.t. } (1-\beta_i)\left(\sum_{j=1}^{n}r_{(ji)1}x_j-b_{i4}\right)+\beta_i\left(\sum_{j=1}^{n}r_{(ji)2}x_j-b_{i3}\right)\geq 0, i=\overline{1,q}\ , \tag{4.13}$$

$$\sum_{j=1}^{n}x_j=1, \tag{4.14}$$

$$M_{1j}\leq x_j\leq M_{2j},\ j=\overline{1,n}. \tag{4.15}$$

This completes the proof. ■

Problem (4.12)-(4.15) is a standard multi-objective linear programming problem. For optimal solution we can used several algorithm of multi-objective programming [12, 15].

For $q\to\infty$ (4.3) we see that $\lim_{q\to\infty}\overline{\widetilde{E}}_w(\widetilde{r})=\frac{r_1+r_2}{2}+\frac{\beta-\alpha}{8}$. Thus we get

COROLLARY 4.1 For $q\to\infty$, *the weighetd possibilistic mean VaR efficient portfolio selection model can be reduce to the following linear programming problem:*

$$\max_{x\in\mathbb{R}^n}\sum_{i=1}^{k}\lambda_i\left[\frac{\sum_{j=1}^{n}r_{(ji)1}x_j+\sum_{j=1}^{n}r_{(ji)2}x_j}{2}+\frac{q\left(\sum_{j=1}^{n}r_{(ji)1}x_j-\sum_{j=1}^{n}r_{(ji)4}x_j\right)}{2(4q-1)}-\sum_{j=1}^{n}c_{ji}x_j\right]$$

$$\text{s.t. } (1-\beta_i)\left(\sum_{j=1}^{n}r_{(ji)1}x_j-b_{i4}\right)+\beta_i\left(\sum_{j=1}^{n}r_{(ji)2}x_j-b_{i3}\right)\geq 0, i=1,...,k$$

$$\sum_{j=1}^{n}x_j=1$$

$$M_{1j}\leq x_j\leq M_{2j},\ j=\overline{1,n}.$$

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